

In the Fourth Dimension

SAINT PAUL AND MILTON SET
FORTH IN GRAPHIC ALGEBRA

NEW HAVEN, March 24. Prof. Andrew W. Phillips, head of the mathematical department at Yale and dean of the graduate school, at a meeting of the Yale Mathematical Club last week presented an outline statement of some considerations in different dimensional spaces, together with the model of a solid which is the projection of a definite figure of four-dimensional space.

For thirty-five years Prof. Phillips has been a teacher at Yale and he introduced on an extensive scale the study of mathematics by the aid of graphic curves and models. Not only in the elementary courses in his department but in more complex models he has made plain to the comprehension of the less advanced students many of the principles of the higher mathematics. Ever since he was a lad he has been delving into mathematics and he has spent years in evolving new models with which to present his ideas to his students.

It is said this new model and the inferences drawn from it will be interesting not only to mathematicians, but equally to others who are students of transcendental problems. It is not surprising to the friends of Prof. Phillips who know his liking for poetry and his familiarity with the Bible to find that his new mathematical demonstration is an interpretation of a passage from Milton and a text of St. Paul.

Prof. Phillips in explaining his new mathematical demonstration made the following statement:

"There is a wonderful fascination to be found in the study of shadows, whether they be of the familiar objects with which we are surrounded, as cast by the sun, the moon or artificial lights, or be produced for practical or scientific or dramatic effects or to teach important lessons of life. Especially is the study exciting when we cannot determine the object which casts the shadow, as when we attempt to find the clouds which are projected by the sunlight on the slopes of the hills or mountains or in the case of the mirage. In the Republic of Plato we find a most gruesome allegory involving shadows, which is narrated by the philosopher to teach an impressive lesson in statecraft, and which also may be applied to teach a lesson concerning the spiritual world."

"And now," I said, "let me show in a figure how far our nature is enlightened or unenlightened. Behold! human beings living in an underground den which has a mouth open toward the light and reaching all along the den; they have been here from their childhood and have their legs and necks chained so that they cannot move and can only see before them. The chains are arranged in such a manner as to prevent them from turning round their heads. Above and behind them the light of a fire is blazing at a distance, and between the fire and the prisoners there is a raised way, and you will see, if you look, a low wall built along the way like the screen which marionette players have in front of them, over which they show their puppets."

"And do you see," I said, "men passing along the wall, some apparently talking and others silent, carrying vessels and statues and figures of animals made of wood and stone and various materials which appear over the wall?"

"You have shown me a strange image and they are strange prisoners."

"Like ourselves," I replied, "and they see only their own shadows or the shadows of one another, which the fire throws on the opposite wall of the cave."

"True," he said, "how could they see anything but the shadows if they were never allowed to move their heads?"

"And of the objects which are being

carried in like manner they would only see the shadows."

"Yes," he said, "and if they were able to talk with one another, would they not suppose that they were naming what was actually before them?"

"And suppose further that the prisoners had an echo which came from the other side, would they not be sure to fancy that the voices which they heard were that of a passing shadow?"

"No question," he replied.

"Beyond question," I said, "the truth of the matter is that they would never get beyond the shadows of images."

"Plato goes on in the dialogue to consider, if one of the prisoners is liberated and is compelled to see the true light and is brought out of the cave and taught to see the objects themselves of which he had seen the shadows, how he would be dazzled and confounded by the sight."

"To take an illustration along the line of the development of this subject in hand, suppose an intelligent creature was confined between two discs and that one of the discs was of glass and contained so that the inhabitant could not see the objects in the world outside but could understand them only by the shadows of them cast on the curtain. Suppose the other disc was so arranged that he could draw all sorts of figures. The interpretation of his mathematical equations and expressions would involve but two dimensions, length and breadth. He might find imaginary roots of his equations and imaginary expressions which he could not interpret because there was no third dimension in his experience."

"The one-dimensional creature, as an eel in a water pipe, could only interpret one-dimensional algebra. It would be plus x and minus x . We in a three-dimensional world may find imaginary quantities as when we plot a sphere and not be able to interpret them in our space. There are many mysteries which we encounter which might be interpreted if we were four-dimensional beings in a four-dimensional space."

"But to bring this down to mathematical form more strictly, let us suppose a one-dimensional creature to inhabit a line world. Assume that he is free to move along his space, the line, and that he can pierce all objects with his line. A circle would to him be manifested by chords of different lengths, and he could explore shadows and sections of two-dimensional space, but only to get lines."

"Suppose next a two-dimensional creature whose dwelling was a mathematical plane and suppose the word shadow to mean to him any manifestation in his plane, a section of a solid, a projection of a figure, as well as a shadow used in its ordinary sense; suppose him to be familiar with algebraic analysis and manipulation; that he could move freely in his plane, that ordinary shadows and projections could be made upon it and that he could pierce all solid objects with his plane. He could not see a sphere, but he could find that its sections were circles by experiment as well as by analysis. He could not see a cube, but he could understand by experiment and analysis that sections made by his plane parallel to a side were squares, that sections parallel to an edge would be rectangles, that oblique sections would be triangles or figures of four, five or six sides."

"Let us now study a definite line of figures."

1. 2. 3.

12 - 0' = 1' + 2' + 3' + 4' + 5' + 6' + 7' + 8' + 9'

"Assuming a square whose side is x ,

suppose this square to grow in length independently of its breadth, making the resulting growth of a rectangle on one side of length x and breadth a . Suppose the original square also to have grown in breadth independently of its length, making a similar rectangle on the top of the square. Figure 2 represents the square with these two growths, and in the equation the square itself is x^2 , and the two additions are $2ax$. If then the two additions grow they will complete the square by the addition of a^2 . (Figure 3)

"Again:

Assuming a cube whose side is x (Figure 4), suppose this cube to grow in length independently of breadth and height, making the resulting slab whose sides are x long and x wide and a thick. Suppose further that the cube grows in breadth the same as it grows in length, we have a similar slab added to the front of the cube. Again, if the cube grows in height as it has grown in length and breadth, a similar slab is added to the top. The result of these additions is Figure 5, and the total of these additions gives $3ax$, the second term on the right hand side of the above equation. Suppose again that these three slabs grow so as to fill in the corners. Each of these growths will be x long and a wide and a thick, the total being $3a^2$, or the third term of the right side of the equation. (See Figure 6.) If now these three corner pieces grow they will complete the Figure 6, and this will be the last term a^3 of the equation."

"Following the method of the projection of the cube, x^3 would be projected as x^2 in our space. The projection should be a four sided figure, the same as the projection of x^2 was a three sided figure. Figure 9 is such a four sided figure, a regular tetrahedron. The fourth dimensional figure should grow in four directions. Hence each side of its projection (Figure 9) should grow a slab represented by ax . It will be seen that each slab of Figure 9 satisfies this expression in projection, x^2 representing the surface of the side and a the thickness. Again if these four slabs grow in projection there will result (see Figure 11) the six corner pieces each represented by x long, a wide and a thick. The projection of $4ax$ would be each represented by a rhomboidal figure, of course side the equal of x in Figure 12. Of course it is understood that the term a^3 is left out of the projection for the reason that a^3 was left out of the projection of the three dimensional cube. We have therefore the expression in projection reduced one degree, and each term multiplied by a constant to account for the obliquity of the projector. Thus $x^2 + 2ax + a^2$ represents the expression for the projection in Figure 12, in three dimensional space, of the fourth degree figure $(x+a)^3$.

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